Laminar-turbulent Transition for Nonisothermal Pipe Flow

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Finite disturbance pipe flow stability criteria are evaluated for constant flux heating of liquids flowing through vertical pipes under small disturbance conditions where velocity profile distortion due to natural convection is significant. For Newtonian flow in long heated pipes the analysis of Hanks correctly predicts the transition to turbulence observed for upflow heating and is qualitatively consistent with the transition to asymmetric flow which occurs in downflow heating, whereas the criterion of Ryan and Johnson predicts stability for experimentally unstable flows. In the practically more important case of short pipe lengths experimental data obtained over the viscosity range from 1 to 21 centipoises show that natural convection induced transition will occur at low Reynolds numbers for both Newtonian and non-Newtonian liquids for conditions consistent with predictions of Hanks' analysis.

No completely general theoretical predictions exist for determining whether flow in a straight, smooth, circular cross section pipe will be laminar or turbulent because of difficulties associated with characterizing the nature of the disturbances. This paper investigates the validity of proposed stability criteria for large disturbance flows by comparing predicted and experimental values of the transition Reynolds number $N_{Re,c}$ for nonisothermal pipe flows in which the interaction of the temperature and velocity fields is significant.

FINITE DISTURBANCE STABILITY CRITERIA

Ryan and Johnson (*10*) proposed the use of a local dimensionless stability parameter:

$$Z_j = -\frac{\rho\, a\, u}{g_c\, \tau_w} \frac{du}{dr} \tag{1}$$

which is a function of the ratio of input energy to energy dissipation for an element of fluid, hypothesizing that when Z_j reaches a maximum value of 808 at some point in a laminar flow field the flow will be sufficiently unstable to become turbulent. The value of 808 was chosen because it leads to a critical Reynolds number (defined with the use of the pipe radius as the length parameter) of 1,050 for an isothermal Newtonian fluid. Ryan and Johnson experimentally verified the stability criterion for isothermal flow of pseudoplastic fluids. Hanks and Christiansen (*7*) extended the criterion to include the heated flow of pseudoplastic liquids, and Hanks (*6*) has applied the criterion to pipe flow of fluids with yield stresses.

Since Z_j is geometry dependent, Hanks (*5*) suggested a more general stability parameter which for steady state rectilinear pipe flows reduces to

$$A = \frac{2\,\rho|u\, du/dr|}{g_c|div\, \tau|} \tag{2}$$

and postulated that when A reaches a critical value of 808 at some point in the flow field, transition will occur. $A = Z_j$ only for fully developed, isothermal Newtonian pipe flows. Equations (1) and (2) are not identical for nonisothermal flows because of fluid property variations with temperature.

Equations (1) and (2) can be used to predict the critical Reynolds number $N_{Re,c}$ above which the flow is unstable by expressing the stresses in terms of velocity derivatives and by substituting the critical values for Z_j and A, respectively. After substitution of dimensionless velocity and position variables and rearrangement, the Ryan and Johnson criterion for fully developed Newtonian flow becomes

$$N_{Re,c} = 808 \frac{(dU/dR)_w}{[U(dU/dR)]_{\max}} \tag{3}$$

and Hanks' generalized criterion becomes

$$N_{Re,c} = 404 \left\{ \frac{|d^2U/dR^2 + (1/R)(dU/dR)|}{|U\, dU/dR|} \right\}_{\min} \tag{4}$$

Although Hanks and Christiansen (*7*) concluded that Equation (1) is a generalized criterion for pipe flows

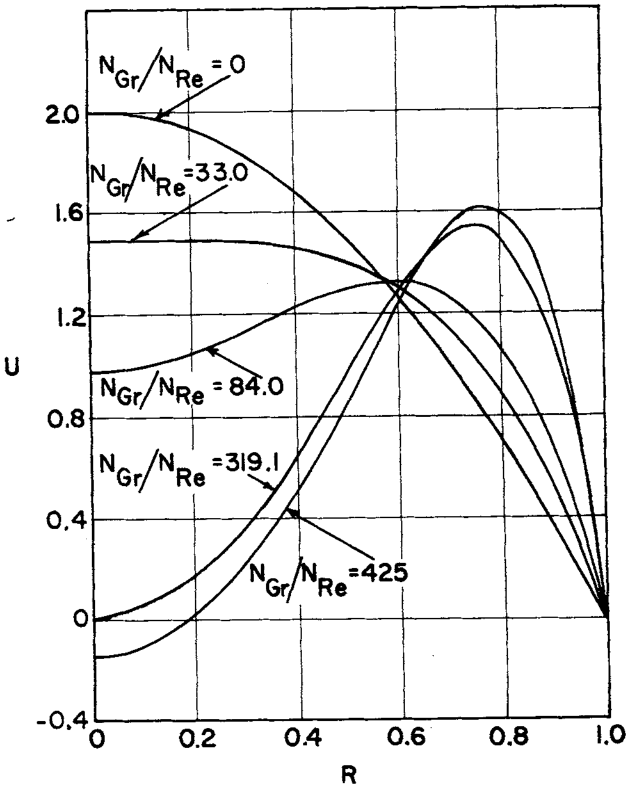

Fig. 1. Velocity profiles for laminar upflow with constant flux heating.

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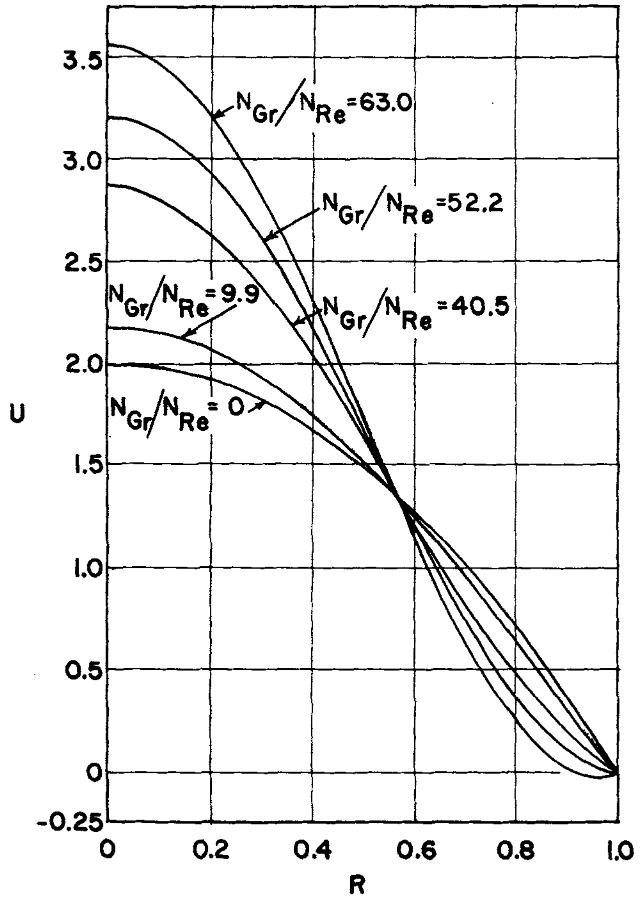

Fig. 2. Velocity profiles for laminar downflow with constant flux heating.

independent of "the shape of the velocity profile or the existence of temperature gradients in the flow field," this conclusion was reached prior to the development of Equation (2) and was based on data obtained for systems where the two equations were similar. Neither equation has been tested for systems where nonisothermal effects significantly distort the velocity profile.

CONSTANT HEAT FLUX VELOCITY FIELDS

For constant flux heating of water flowing through a vertical pipe, experimental investigations have shown that natural convection effects can significantly distort the laminar flow field (*8*). This system is a good one for evaluating the stability criteria, because analytical solutions for the fully developed distorted laminar velocity and temperature profiles for Newtonian liquids have been obtained by several investigators (*3, 8, 9*).

Theoretical velocity profiles are shown in Figures 1 and 2. The profile distortion is a function of the ratio N_{Gr}/N_{Re} of Grashof and radius Reynolds numbers, and it also depends on whether the natural convection near the pipe wall is in the direction of or opposite to the direction of the forced flow.

For upflow heating the dimensionless velocity distribution is given by

$$U = \frac{K^{1/4}[bei(K^{1/4})ber(K^{1/4}R) - ber(K^{1/4})bei(K^{1/4}R)]}{2[ber'(K^{1/4})ber(K^{1/4}) + bei'(K^{1/4})bei(K^{1/4})]} \quad (5)$$

where

$$K = 2\,(N_{Gr}/N_{Re}) \quad (6)$$

For $N_{Gr}/N_{Re} > 32.94$ the maximum local velocity no longer occurs at the center of the pipe, and for $N_{Gr}/N_{Re} > 319.1$ flow reversal is predicted.

For downflow heating

$$U = \frac{K^{1/4}[I_0(K^{1/4})J_0(K^{1/4}R) - J_0(K^{1/4})I_0(K^{1/4}R)]}{2[J_1(K^{1/4})I_0(K^{1/4}) - J_0(K^{1/4})I_1(K^{1/4})]} \quad (7)$$

Above $N_{Gr}/N_{Re} = 9.87$ the velocity profiles possess a point of inflection, and flow reversal at the wall is predicted for $N_{Gr}/N_{Re} > 52.2$.

No theoretical analysis exists for the fully developed distorted velocity profiles obtained for constant flux heating of non-Newtonian fluids under conditions where distortion is primarily caused by convection effects. However, for power law liquids dimensional analysis of the equations of motion indicates the flow fields should be a function only of a generalized Grashof-Reynolds number ratio N'_{Gr}/N'_{Re} and power law exponent n (*2*).

STABILITY RESULTS

Critical Reynolds numbers have been predicted by using Equations (3) and (4) for the velocity distributions given by Equations (5) and (7). Since the fully developed distorted flow fields are unique functions of N_{Gr}/N_{Re}, $N_{Re,c}$ is also a function only of this parameter. Calculated values of $N_{Re,c}$ are shown in Figures 3 and 4 for upflow and downflow heating, respectively, and the theoretical values of N_{Gr}/N_{Re} above which flow reversal is predicted are also indicated.

The stability of constant heat flux distorted flow fields has been studied experimentally under conditions such that any instability observed resulted from initially small disturbances (*12, 13*). In heated pipes long enough to establish fully developed flow it was found experimentally that transition to a disturbed flow was a function of the shape of the distorted velocity profile, independent of flow rate over a wide range of Reynolds numbers. For upflow heating transition occurred for values of $N_{Gr}/N_{Re} \simeq 33$ even for N_{Re} as low as 50, indicating that as soon as the velocity profile develops a dimple at the pipe center the flow field becomes unstable even at low flow rates. For downflow heating transition to unsteady flow was associated with flow separation at the pipe wall, instability occurring for values of N_{Gr}/N_{Re} slightly greater than 52 for N_{Re} from 50 to 4,000, and asymmetry of the flow field was observed at even lower values of N_{Gr}/N_{Re}. Transition values of N_{Gr}/N_{Re} are shown in Figures 3 and 4.

The convection instabilities previously studied experimentally were all observed with water as the liquid in pipes at least 114 pipe diameters in length. To answer the questions of whether such instabilities occur in more

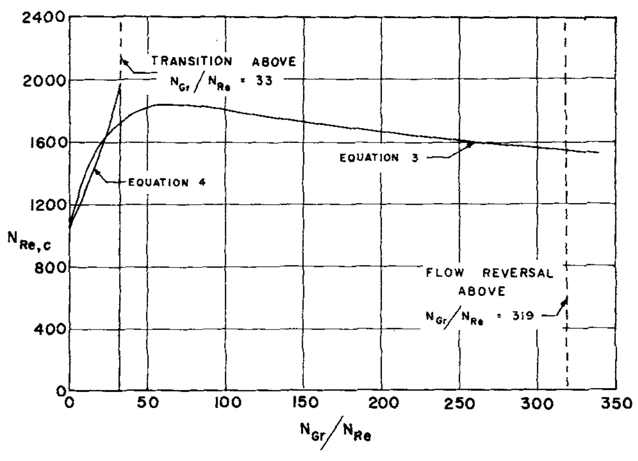

Fig. 3. Predicted transition Reynolds number for upflow heating.

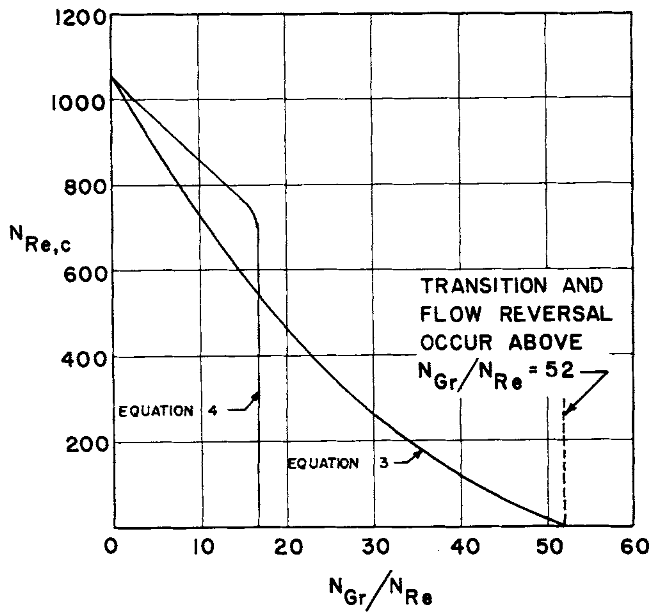

Fig. 4. Predicted transition Reynolds number for downflow heating.

practically important short lengths of pipe and whether they occur for liquids more viscous than water, the stability of more viscous Newtonian and non-Newtonian fluids was investigated for upflow in a 2½-ft. heated length of 1-in. I.D. copper pipe in which the flow attained its fully developed isothermal velocity distribution prior to entering the heat transfer section. The flow system used was similar to that described in the literature (*12*), and transition was observed by detecting temperature fluctuations in the flow field.

In all experiments convection induced transition to fluctuating flow was observed at low Reynolds numbers. Fluid properties and transition values are summarized in Table 1. The nature of the equipment was such that for each fluid transition could be studied over only a very limited Reynolds number range because of heat input limitations. The progressively lower Reynolds numbers studied with increasing liquid viscosity reflect this limitation rather than any effect of viscosity on the transition Reynolds numbers, and the data serve merely to indicate the occurrence of transition at low flow rates in short heated pipes.

DISCUSSION

Upflow Heating

For values of N_{Gr}/N_{Re} less than 33, where predicted values of $N_{Re,c}$ are smaller than values of N_{Re} at which Scheele and Hanratty (*12*) observed stable flow in small disturbance experiments, there is no inconsistency between theory and experiment, since transition would be expected at a lower N_{Re} in large disturbance situations. No data exist for evaluating the validity of the stabilizing effect predicted by both stability criteria for moderate profile distortion. However, above $N_{Gr}/N_{Re} = 33$ the Hanks stability criterion correctly predicts the instability experimentally observed, whereas the Ryan and Johnson equation is clearly incorrect, predicting stability for $N_{Re} \lesssim 1{,}500$ even for velocity profiles where there is flow reversal in the center of the pipe.

The reason the Hanks criterion works so well in predicting the observed instability, even though it is a large disturbance criterion being applied to small disturbance phenomena, may be seen by considering the inviscid instability criterion for axisymmetric flows. For nonrotationally symmetric disturbances, Batchelor and Gill (*1*) and Schade (*11*) show that for instability to occur

$$\frac{d^2U}{dR^2} + \frac{1}{R}\,\frac{(f^2 - \alpha^2 R^2)}{(f^2 + \alpha^2 R^2)}\,\frac{dU}{dR} = 0 \qquad (8)$$

In the limit as the wave number α of the disturbance approaches 0, Equation (8) reduces to

$$\frac{d^2U}{dR^2} + \frac{1}{R}\,\frac{dU}{dR} = 0 \qquad (9)$$

Equation (4) predicts $N_{Re,c} = 0$ when Equation (9) is satisfied and is thus consistent with a special case of hydrodynamic stability theory for small disturbance flows. The inviscid instability of these flows will be considered more fully in a future paper.

The results in Table 1 indicate that convection induced transition can be of importance both in short lengths of pipe and for fluids more viscous than water. Large values of N_{Gr}/N_{Re} are required to cause transition because the 2½-ft. heated length of pipe is insufficient to produce fully developed distorted flow fields. Despite the rather high transition ratios, the heating conditions are still quite mild, wall temperatures 3.0° to 8.1°C. greater than the average fluid temperatures at the outlet of the heat transfer section producing fluctuating flow within thirty pipe diameters for average fluid velocities of 0.6 to 1.2 cm./sec. A comparison of theory with these experimental results cannot be rigorously made since equations do not exist for the developing profiles as a function of heated pipe length for either the Newtonian or the non-Newtonian liquids. However, for the Newtonian solutions with initially parabolic velocity distributions the changes in the developing profiles with heated length should be similar to the changes in fully developed profiles with increasing N_{Gr}/N_{Re}, which makes it likely that the stability criterion of Hanks would predict the low transition Reynolds numbers listed in Table 1 if expressions did exist for the developing profiles.

Downflow Heating

For low values of N_{Gr}/N_{Re} experimental results for small disturbance flows do not show $N_{Re,c}$ decreasing with increasing N_{Gr}/N_{Re} as shown in Figure 4. As in the upflow experiments this apparent disagreement does not invalidate theory because the predicted critical Reynolds numbers for large disturbance flows are, in all instances, smaller than values of N_{Re} for which laminar flow fields have been observed in small disturbance experiments.

It would appear that in contrast to the upflow results, the transition to turbulence observed at all Reynolds numbers for fully developed flows for $N_{Gr}/N_{Re} > 52$ is predicted correctly by Equation (3) and not by Equation (4). However, experimental heat transfer measurements (*4, 12*) and visual observations in a constant wall temperature system (*13*) indicate asymmetric laminar flow can exist in downflow experiments. No data exist on the transition to asymmetric flow, but it occurs at values of N_{Gr}/N_{Re} considerably smaller than 52, and it may be that the transition to asymmetric flow is predicted by Hanks' analysis which gives $N_{Re,c} = 0$ for $N_{Gr}/N_{Re} > 16.84$. Since Hanks' criterion is only a special case of the more general inviscid stability theory, quantitative agreement between theory and experiment may not result. The criterion of Ryan and Johnson would not predict this transition which, while it does not result in turbulent flow, does affect the transport processes.

CONCLUSIONS

The pipe flow stability criterion of Hanks, Equations (2) and (4), is consistent with experimental studies in

TABLE 1. TRANSITION RESULTS FOR CONSTANT FLUX UPFLOW HEATING IN A VERTICAL PIPE—HEATED $l/2a = 30$

Liquid	$\mu \times 10^2$, g./(cm.)(sec.)		U_{avg}, cm./sec.	Outlet $t_w - t_{avg}$, °C.	N_{Re}	N_{Gr}/N_{Re}
Water	0.89		1.21	3.02	169	767
45 wt. % sucrose-water	8.4		0.88	5.82	14.2	286
	8.4		1.12	7.73	18.6	276
55 wt. % sucrose-water	21.5		0.55	5.98	5.1	247
	21.5		0.76	8.09	7.1	249
	$m \times 10^2$, g./(cm.)(sec.)$^{2-n}$	n				
0.11 wt. % Polyox WSR 301-water	7.6	0.85	0.73	4.42	10.9	95
	7.6	0.85	1.10	7.45	18.0	117

which large flow field distortions result from natural convection effects. For constant flux heating of Newtonian liquids, the criterion correctly predicts the observed transition to turbulence for upflows, and the transition to asymmetric flow detected at low values of N_{Gr}/N_{Re} for downflows shows qualitative agreement with theory. Transition is predicted incorrectly by the Ryan and Johnson criterion, Equations (1) and (3), instability occurring in some cases for predicted stable flows. The Hanks analysis agrees better with the data because it more correctly accounts for the shape of the flow field on stability and is, in fact, consistent with a limiting case of inviscid stability theory.

Natural convection induced instability, previously observed for water, also occurs for more viscous Newtonian and non-Newtonian liquids. Convection effects are most likely to be negligible for high viscosity, high Reynolds number flows since the heat flux per unit length required to produce a given value of N_{Gr}/N_{Re} is proportional to $U_{avg}\mu$ and significant profile distortion may require impractically large heat inputs. In lengths of pipe short enough to be of practical interest the velocity profiles may still be developing, in which case critical values of N_{Gr}/N_{Re} are greater than for fully developed flows. These results are apparently consistent with Hanks' criterion, but conclusive proof will require the development of equations for the developing profiles.

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NOTATION

a = pipe radius
A = stability parameter defined by Equation (2)
$ber\ (x)$ = real part of $I_o(\sqrt{i}\ x)$
$bei\ (x)$ = imaginary part of $I_o(\sqrt{i}\ x)$
$ber'\ (x)$ = $d\ ber\ (x)/dx$
$bei'\ (x)$ = $d\ bei\ (x)/dx$
f = disturbance parameter, 0 for rotationally symmetric disturbances
g = acceleration of gravity
g_c = force-mass conversion factor
$I_o(x)$ = modified Bessel function of first kind of order zero
$I_1(x)$ = modified Bessel function of first kind of order one
$J_o(x)$ = Bessel function of first kind of order zero
$J_1(x)$ = Bessel function of first kind of order one
k = liquid thermal conductivity
K = dimensionless parameter defined by Equation (6)
l = heated length of pipe
m = consistency index for power law fluid
n = power law exponent or flow behavior index
N_{Gr} = modified Grashof number, $a^4\rho^2 g\beta q/k\mu^2$
N'_{Gr} = generalized modified Grashof number, $a^4\rho^2 g\beta q/k\mu^2_{eff}$
N_{Re} = Reynolds number, $a\ U_{avg}\rho/\mu$
$N_{Re,c}$ = critical Reynolds number
N'_{Re} = generalized Reynolds number, $a\ U_{avg}\rho/\mu_{eff}$
q = heat input to liquid per unit surface area
r = radial position coordinate
R = dimensionless radius, r/a
u = local axial velocity
U = dimensionless velocity, u/U_{avg}
U_{avg} = average fluid velocity in axial direction
w = subscript denoting quantity evaluated at $R = 1$
Z_j = stability parameter defined by Equation (1)

Greek Letters

α = disturbance wave number
β = coefficient of volumetric expansion
μ = liquid viscosity
μ_{eff} = effective viscosity for power law fluid, $m(U_{avg}/a)^{n-1}$
ρ = liquid density
τ = stress tensor
τ_w = wall shear stress

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